BEREZIN INTEGRALS AND POISSON PROCESSES

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Dedicated to the memory of Michel Sirugue

Abstract. We show that the calculation of Berezin integrals over anticommuting variables can be reduced to the evaluation of expectations of functionals of Poisson processes via an appropriate Feynman-Kac formula. In this way the tools of ordinary analysis can be applied to Berezin integrals and, as an example, we prove a simple upper bound. Possible applications of our results are briefly mentioned.

Key words: Grassmann algebra, Feynman-Kac formula, Dichotomic variables, Poisson process.

1. Introduction During the last decades the functional integral has become the standard approach to the quantization of systems with infinitely many degrees of freedom. In typical cases like QED and QCD which involve both bosons and fermions the integral has to deal with anticommuting variables belonging to a Grassmann algebra and a lucid exposition of the rules of integration over these variables can be found in [B1], [FS].

With the discovery of supersymmetry (SUSY) anticommutative integration has received further impetus and has been applied in different areas of physics and mathematics. SUSY has been first introduced in particle physics to express a possible fundamental symmetry between bosons and fermions and then has found several applications as a formal tool in the theory of complex systems like heavy nuclei and more recently disordered or chaotic

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mesoscopic systems [VWZ, F, Z]. In mathematics it plays a role in various approaches to index theorems and other topics of differential geometry [BZ].

In physical applications of SUSY one introduces fields Ψ which have two types of components, boson-like $\varphi_1, ..., \varphi_{2n}$ which are just ordinary numerical variables and fermi-like $\xi_1, ..., \xi_{2n}$ where the ξ-s are anticommuting Grassmann variables. Supersymmetry transformations mix boson and fermion components.

One is usually interested in the calculation of formal expressions like

$$\int d[\phi]d[\xi] \prod_i \varphi_i \prod_j \xi_j exp(-S(\Psi))$$

called correlation functions. $S(\Psi)$ is a functional invariant under supersymmetry transformations.

Integrals over anticommuting variables were introduced by Berezin also in statistical mechanics to represent the partition function of the planar Ising model [B2] and the generating function of related combinatorial problems [B3]. See also [RZ].

The rules of calculation of these integrals, known as Berezin integrals, are recalled in the next section. In spite of the fact that this formalism allows compact and very powerful manipulations of the expressions of interest and in some cases exact calculations, it has the drawback that the usual tools of analysis are not applicable to the anticommuting variables. In particular one cannot easily find bounds for anticommutative integrals.

The purpose of this paper is to show that the any Berezin integral can be represented in terms of the expectations of appropriate functionals of Poisson processes. On the basis of this representation ordinary analysis can be used and in particular upper bounds can be obtained. Furthermore correlation functions as above can be expressed entirely as expectations over ordinary stochastic variables.

The starting point of our analysis is a generalized Feynman-Kac formula developed in the early eighties to express the solutions of the imaginary time Pauli equation [DJLS]. The equation is

$$\partial_t \psi_t = -\frac{1}{2}(-i\nabla - \mathbf{A}(x))^2 \psi_t - V(x)\psi_t + \frac{1}{2}\mathbf{H}(x) \cdot \boldsymbol{\sigma} \psi_t \tag{1.1}$$

$\mathbf{A}$ and $\mathbf{H}$ are the vector potential and the magnetic field respectively. $\boldsymbol{\sigma}$ denotes the Pauli matrices in the usual representation. In [DJLS] we proved that the initial value problem

is solved by

$$\psi_t(x,\sigma) = e^t\,\mathbf{E}\big[\psi_0(x+w_t,(-)^{N_t}\sigma)\exp\big(-\int_0^t V(x+w_\tau)d\tau - i\int_0^t \mathbf{A}(x+w_\tau)\cdot d\mathbf{w}_\tau +$$

$$\frac{1}{2}\int_0^t H_z(x+w_\tau)(-)^{N_\tau}\sigma d\tau + \int_0^t \log[\frac{1}{2}(H_x(x+w_\tau) - i(-)^{N_\tau}\sigma H_y(x+w_\tau))]dN_\tau\big)\big] \tag{1.2}$$

σ is a dichotomic variable which can take the values ± 1. The expectation is taken with respect to the Wiener process w_t and to the Poisson process N_t. For the understanding of this formula we have to explain the meaning of the stochastic integral $\int dN_t$. A Poisson process is a jump process characterized by the following probabilities

$$P(N_{t+\Delta t} - N_t = k) = \frac{(\Delta t)^k}{k!}e^{-\Delta t}$$

Its trajectories are therefore piecewise constant increasing functions and we shall assume that at each jump they are continuous from the left. The stochastic integral is just an ordinary Stieltjes integral

$$\int_0^t f(\tau)dN_\tau = \sum_1^n f(\tau_i)$$

where τ_i are random jump times in the interval $[0,t)$ which are distributed exponentially, that is $P(\tau \le t) = 1 - e^{-t}$.

The interesting property of formula (1.2) is that by letting the spinor indices to become a stochastic process the Pauli matrices have disappeared from the expression of the evolution operator and their algebra is taken into account completely by the expectation over the jump process. The power of the approach was demonstrated by proving a non trivial paramagnetic inequality which shows that in three dimensions the evolution is bounded above by the evolution in a magnetic field which lies in a plane and whose components are simply related to the original magnetic field. This is easily seen by taking absolute values and implies for the ground states

$$E_0(0,0,0;(H_1^2+H_2^2)^{1/2},0,H_3) \le E_0(A_1,A_2,A_3;H_1,H_2,H_3)$$

For a recent application of the [DJLS] approach see [ER].

Since the Pauli matrices are objects which belong to a Clifford algebra the above findings suggested a possible connection between calculus with Poisson processes and calculus with anticommuting variables. It is the main purpose of this paper to implement such a connection. To illustrate how the connection comes about in Sec. 3 we take up again the case of Pauli type equations and we observe that they can be interpreted as evolution equations over a Grassmann algebra. This type of evolution was considered for instance by Berezin and Marinov [BM], [M]. Solutions of evolution equations over Grassmann algebras can be expressed as Berezin integrals which represent the convolution of the kernel of the evolution operator with the initial condition. A straightforward comparison with the solution given in [DJLS] provides the identification of the anticommutative integrals with the appropriate Poisson expectations.

In order to develop the theory in a systematic way for an arbitrary but finite number of anticommutative variables in Sec. 4 we introduce a representation of Grassmann algebras in the space of functions of dichotomic variables that we call the σ-representation. This space was used by Wigner in his book on group theory [W] to find the representations of the symmetric group connected with the exclusion principle.

In Sec. 5 we develop the necessary theory of semigroups associated with Poisson processes and in Sec. 6 we make the identification of Berezin integrals with appropriate expectations. We also derive a general inequality.

In Sec. 7 we discuss in particular the Gaussian anticommutative integrals due to their importance in physical applications.

We conclude this introduction with some comments on possible interesting applications of the results obtained in this paper. The semigroups associated to Poisson processes encompass all Hamiltonian semigroups $\{\exp(-tH)\}_{t\geq 0}$ for k interacting $\frac{1}{2}$-spins in an external magnetic field as, for instance, a Heisenberg ferromagnet or, by a slight change of language, any Hamiltonian semigroup for models describing interacting fermions on a finite lattice. Our representation can therefore be used both for theoretical or simulation studies of the statistical mechanics of such systems.

A particularly interesting case to which our approach can be applied is the calculation of the Dirac propagator on a lattice, an important problem in the study of QCD on the lattice. Presumably in this way it is possible to obtain a simplification of the methods used at present in simulations. See the remark at the end of section 7.

More generally in all cases where SUSY is relevant our approach may be a useful tool.

We believe that our results have also an independent interest in so far as they establish a direct connection between algebraic objects as those represented by Berezin integrals and analytic expressions.

As a final remark we observe that the Wiener process and the Poisson process are both Levy processes and are actually the two limit cases of the Levy-Khinchin formula. We find therefore quite satisfying their correspondence at the euclidean level with the two basic types of particles in nature, bosons and fermions. Then the question naturally arises: do the other processes described by the Levy-Khinchin formula have any relevance for physics?

2. Analysis on Grassmann Algebras

We open this Section by a short review of standard definitions and results and refer reader to [B], [D], [FS] for more detailed information.

By $\mathbf{G}(k)$ we denote the Grassmann algebra over $\mathbb{C}$ generated by its identity $\mathbf{1}_k$ and a family $\{\xi_1, \ldots, \xi_k\}$ of generators which obey the following commutation relations:

$$\xi_i \xi_j = -\xi_j \xi_i \, ; \quad \forall\, i, j. \tag{2.1}$$

For future reference, $\mathbf{G}^n(k) \simeq \mathbf{G}(nk)$ will be the Grassmann algebra generated by $\mathbf{1}_k^n$ and $\{\xi_1^1, \ldots, \xi_k^1, \xi_1^2, \ldots, \xi_k^2, \ldots, \xi_1^n, \ldots, \xi_k^n\}$, and by convention, $\mathbf{G}^1(k) = \mathbf{G}(k)$.

The collection $\{1, \ldots, k\}$ of labels is any non - empty finite set endowed with a total ordering.

Elements of $\mathbf{G}$(k) of the form $\xi_{i_1}\xi_{i_2} \cdot \ldots \cdot \xi_{i_n}$ are called monomials; we will use the set of ordered multiindices $M_k = \{\mu = (\mu_1, \ldots, \mu_n) \,:\, 1 \leq \mu_1 < \mu_2 < \ldots < \mu_n \leq k\}$, and write:

$$\xi^\mu = \xi_{\mu_1} \cdot \ldots \cdot \xi_{\mu_n} \, ,$$

for $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. As a linear space $\mathbf{G}$(k) has dimension 2^k, and each element $F(\xi) \in \mathbf{G}$(k) can be represented in a unique way as a polynomial with complex coefficients:

$$F(\xi) = F(\xi) = f_0 \cdot \mathbf{1}_k + \sum_{r=1}^{k} \sum_{1 \leq i_1 < \ldots < i_r \leq k} f_{i_1, \ldots, i_k} \xi_{i_1} \cdot \ldots \cdot \xi_{i_r} = \sum_{\mu \in M_n} f_\mu \cdot \xi^\mu \, , \tag{2.2}$$

where $f_\mu \in \mathbb{C}$, and therefore $\mathbf{G}(k)$ is naturally graded. It is advantageous thinking of $F(\xi)$ as a "function" of the Grassmann "variables" $\xi_1, \ldots, \xi_k$ that is to say of the "Fermi field" $\{\xi_1, \ldots, \xi_k\}$.

Analysis over $\mathbf{G}(k)$ is based upon left derivatives $\frac{\delta}{\delta\xi_1}, \dots, \frac{\delta}{\delta\xi_k}$ and Berezin integral, which are defined as follows:

$$\frac{\delta}{\delta\xi_i}\xi_{\mu_1}\cdot\ldots\cdot\xi_{\mu_n} \stackrel{\text{def}}{=} \delta_{\mu_1 i}\cdot\xi_{\mu_2}\cdot\ldots\cdot\xi_{\mu_n} - \delta_{\mu_2 i}\cdot\xi_{\mu_1}\xi_{\mu_3}\cdot\ldots\cdot\xi_{\mu_n} + (-1)^{k-1}\delta_{\mu_k i}\cdot\xi_{\mu_1}\cdot\ldots\cdot\xi_{\mu_{k-1}} =$$

$$\sum_{j=1}^{k}(-1)^{j-1}\delta_{\mu_j i}\cdot\xi_{\mu_1}\cdot\ldots\cancel{\xi_{\mu_j}}\cdot\ldots\cdot\xi_{\mu_{k-1}}\,,$$

where the sign / over the generator ξ_{μ_j} means that it is omitted.

To define an integral we introduce symbols $d\xi_1, \dots, d\xi_k$ satisfying the following commutation relations:

$$\{d\xi_i, d\xi_j\} = \{d\xi_i, \xi_j\} = 0;$$

where $\{a, b\} = a\cdot b + b\cdot a$, and define "basic" integrals:

$$\int^B d\xi_i = 0, \qquad \int^B \xi_\mu\cdot\xi_i\, d\xi_i = \xi_\mu\,;$$

if $\mu_j \neq i$ for all j in μ. We extend the integral on $\mathbf{G}$(k) by linearity and call it the Berezin integral. In general

$$\int^B F(\xi)d\xi_k\cdot\ldots\cdot d\xi_1 \stackrel{\text{def}}{=} \int^B F(\xi)\mathcal{D}_k\xi \,\equiv\, f_{1,\dots,k} \in \mathbb{C}\,, \tag{2.3}$$

and observe that up to a multiplicative constant, is uniquely defined as the only linear form over $\mathbf{G}$(k) alternating under permutations of the Grassmannian variables. It transforms [B] as

$$\int^B F(R\xi)\mathcal{D}_k\xi \,=\, (\det R)\cdot\int^B F(\xi)\mathcal{D}_k\xi\,, \tag{2.4}$$

under the linear substitution $\xi_i \,\to\, \sum_{j=1}^k R_{ij}\xi_j$.

A specially important case is provided by "Gaussian integrals":

$$\int^B \exp\{\frac{1}{2}\sum_{i,j=1}^{k} A_{ij}\xi_i\xi_j\}\mathcal{D}_k\xi\,; \tag{2.5}$$

where one can always assume that A = (A_{ij}) is an antisymmetric matrix, otherwise it could be replaced by 2^{-1}(A - A^T). "Gaussian integrals" vanish for odd k while, by exploiting (2.4), we get

$$\int^B \exp\{\sum_{1\le h<k\le 2k} A_{hk}\xi_h\xi_k\}\mathcal{D}_{2k}\xi \;=\; Pf\,A\,; \tag{2.6}$$

The Pfaffian Pf A of the triangular array $\{A_{hk}\}_{1\le h<k\le 2k}$ is defined by :

$$PfA \;=\; \sum_{\pi}(-1)^{\pi}A_{i_1j_1}\;\cdot\ldots\cdot A_{i_kj_k}\,; \tag{2.7}$$

where the sum $\sum_\pi$ is taken over all $(2k)!/2^k k!$ ways of pairing of the elements of the set $\{1,\,2,\ldots,2k\}$ where $(-1)^\pi$ is the parity of the permutation $\pi \;=\; (i_1j_1,\ldots,i_kj_k)$.

Let us consider the operator of left derivative $\frac{\delta}{\delta\xi_i}$ and the operator $\widehat{\xi_i}$ of left multiplication by the element ξ_i, both acting on $\mathbf{G}$(k) (see [B]). In the sequel we shall omit the hat if confusion does not arise.

We recall that all linear operators acting on $\mathbf{G}(k)$ belong to the Clifford (or Spinor) algebra $\mathbf{C}$(2k) generated by its identity $\hat{\mathbf{1}}_{2k}$ and the operators $\frac{\delta}{\delta\xi_1},\ldots,\frac{\delta}{\delta\xi_k}$, and $\xi_1,\ldots,\xi_k$, which satisfy the following commutation relations:

$$\left\{\xi_i,\,\frac{\delta}{\delta\xi_j}\right\} \;=\; \delta_{ij}\,, \qquad \left\{\xi_i,\,\xi_j\right\} \;=\; \left\{\frac{\delta}{\delta\xi_i},\,\frac{\delta}{\delta\xi_j}\right\} \;=\; 0\,, \qquad i,j \;=\; 1,\ldots,k.$$

The algebra $\mathbf{C}$(2k) is isomorphic to the CAR algebra [BR] over a 2^k - dimensional Hilbert space since operators $\frac{\delta}{\delta\xi_i}$ and ξ_i might be interpreted as annihilation and creation operators for a Fermi system with k degrees of freedom.

We end the Section by defining kernels of operators acting on $\mathbf{G}$(k). We recall that to each $L\,\in\,\mathbf{C}$(2k) corresponds a unique element $\mathrm{Ker}(L)(\xi,\xi')$ of the Grassmann algebra $\mathbf{G}$(2k), generated by $\xi_1,\ldots,\xi_k$, and ${\xi'}_1,\ldots,{\xi'}_k$, such that

$$(LF)(\xi) \;=\; \int_B \mathrm{Ker}(L)(\xi,\xi')\cdot F(\xi')\;\mathcal{D}_k\xi'\,,$$

3. Evolution on $\mathbf{G}(k)$

Let us consider the evolution given by the equation:

$$\frac{\partial f_t}{\partial t} = L\, f_t\,; \tag{3.1}$$

where $f_t \in \mathbf{G}(k)$ and $L \in \mathbf{C}$(2k).

In this section first we solve the equation (3.1) by constructing a kernel for the operator $\exp(tL)$ using standard tools of Grassmannian analysis and writing the solution of (3.1) as an element of $\mathbf{G}(k)$ with coefficients given by certain Berezin integrals, which we compute explicitely, and second, we show that these integrals could be represented as expectations with respect to a properly chosen family of standard Poisson processes.

We introduce some necessary constructions related to the formal description of continuous time evolution on $\mathbf{G}(k)$. In order to do that it is convenient to embed $\mathbf{G}$(k) into an extended Grassmann algebra $\mathbf{G}_\infty(k)$, and we will proceed in the following way (cf [MIS]). Consider three Grassmann algebras:

the Grassmann algebra $\Gamma_\tau(k)$, $\tau = (t_1, \dots, t_m)$, generated by:

$$\mathbf{1},\ \xi_1(t_i), \dots, \xi_k(t_i);\ \rho_1(t_i), \dots, \rho_k(t_i); \qquad i = 1, \dots, m;$$

the Grassmann algebra $\Gamma_{t]}(k)$ generated by:

$$\mathbf{1},\ \xi_1(s), \dots, \xi_k(s);\ \rho_1(s), \dots, \rho_k(s); \qquad s \le t;$$

and the Grassmann algebra $\Gamma_\infty(k)$ generated by:

$$\mathbf{1},\ \xi_1(s), \dots, \xi_k(s);\ \rho_1(s), \dots, \rho_k(s); \qquad s > 0;$$

where

$$\xi_i(s)\cdot\xi_j(t) \ =\ -\,\xi_j(t)\cdot\xi_i(s);\ \rho_i(s)\cdot\rho_j(t) \ =\ -\,\rho_j(t)\cdot\rho_i(s),$$

and

$$\rho_i(s)\cdot\xi_j(t) \ =\ -\,\xi_j(t)\cdot\rho_i(s) \quad \forall\, i,j, \ \ \forall\, s,t > 0.$$

We consider exterior algebras:

$$\begin{aligned} \mathbf{G}_\tau(k) &= \mathbf{G}_0(k) \bigwedge \Gamma_\tau(k);\\ \mathbf{G}_{t\,]}(k) &= \mathbf{G}_0(k) \bigwedge \Gamma_{t\,]}(k);\\ \mathbf{G}_\infty(k) &= \mathbf{G}_0(k) \bigwedge \Gamma_\infty(k); \end{aligned}$$

where $\mathbf{G}_0(k) = \mathbf{G}$(k).

Remark. a) For the construction of Grassmann algebras with an infinite number of generators we refer also to [B], [R], [S]. b) The generators ρ's will be used only for "Fourier transform" - type expressions and play only an auxilliary role here.

Example. The Pauli equation on $\mathbf{G}(1)$.

Let us consider evolution on $\mathbf{G}(1)$, generated by $\mathbf{1}$ and ξ_1, given by (3.1) where

$$L = h_1\,\sigma_1 + h_2\,\sigma_2 + h_3\,\sigma_3\,;$$

and we identify:

$$\sigma_1 = \xi + \frac{\delta}{\delta\xi}\,; \qquad \sigma_2 = i\left(\xi - \frac{\delta}{\delta\xi}\right); \qquad \sigma_3 = \frac{1}{i}\sigma_1\sigma_2\,;$$

so that the usual commutation rules of Pauli matrices are satisfied.

As it will be proven in Theorem (3.1) the kernel of the operator $exp(tL)$ can be written as the limit:

$$\mathrm{Ker}(e^{tL})(\xi,\xi') = \lim_{m\to\infty} Q_t^m(\xi,\xi');$$

where

$$Q_t^m(\xi,\xi') = \underbrace{\int^B \cdot \ldots \cdot \int^B}_{m-\text{times}} P_t^m(\xi,\xi_t,\rho_t,\xi')\mathcal{D}_m\rho_t\,\mathcal{D}_m\xi_t\,;$$

which is the kernel of the operator $(\mathbf{1} + \frac{t}{m}L)^m$ and $P_t^m(\xi,\xi_t,\rho_t,\xi') \in \mathbf{G}_0(1)\bigwedge\Gamma_{\tau(t,m)}(1)$, with $\tau(t,m) = (t/m,\ 2t/m,\ \ldots,[(m-1)t]/m,\ t)$, and has the following form:

$$P_t^m(\xi,\xi_t,\rho_t,\xi') =$$

$$= \exp\Bigg\{\sum_{j=0}^{m}\frac{t}{m}\Bigg\{h_1\left[\xi\left(\frac{j\cdot t}{m}\right)+\rho\left(\frac{j\cdot t}{m}\right)\right] + ih_2\left[\xi\left(\frac{j\cdot t}{m}\right)-\rho\left(\frac{j\cdot t}{m}\right)\right] +$$

$$+h_3\left[1-2\xi\left(\frac{j\cdot t}{m}\right)\rho\left(\frac{j\cdot t}{m}\right)\right] - \rho\left(\frac{j\cdot t}{m}\right)\left[\xi\left(\frac{j\cdot t}{m}\right)+\xi\left(\frac{(j-1)t}{m}\right)\right]\Bigg\}\Bigg\};$$

where

$$\mathcal{D}_m\rho_t = d\rho\left(\frac{t}{m}\right)\cdot\ldots\cdot d\rho(t);$$

$$\mathcal{D}_m\xi_t = d\xi\left(\frac{t}{m}\right)\cdot\ldots\cdot d\xi\left(\frac{(m-1)t}{m}\right);$$

and we put $\xi(0) = \xi'$, $\xi(t) = \xi$. In this way the solution of Equation (3.1) with initial data $F(\xi) = f_0 \cdot \mathbf{1} + f_1 \cdot \xi$; f_0, $f_1 \in \mathbb{C}$ can be written:

$$F_t(\xi) = (e^{tL}\, F)(\xi) = \int^B \left[\mathrm{Ker}(e^{tL})(\xi, \xi') F(\xi')\right] d\xi'; \tag{3.2}$$

and we have:

$$F_t(\xi) = f_0(t)\mathbf{1} \; + \; f_1(t) \cdot \xi \,,$$

where

$$f_0(t) = -\int^B \int^B \left[\mathrm{Ker}(e^{tL})(\xi, \xi') F(\xi') \cdot \xi\right] \, d\xi' \, d\xi \tag{3.3}$$

$$f_1(t) = \int^B \int^B \left[\mathrm{Ker}(e^{tL})(\xi, \xi') F(\xi')\right] d\xi' \, d\xi. \tag{3.4}$$

On the other hand from equation (1.2) specialized to the present simplified case we have that:

$$f_0(t) = e^t\, \mathbf{E}\big[f_{\frac{1-(-1)^{N_t}}{2}}$$

$$\cdot \exp\Big(\int_0^t \ell n(h_1 - i(-1)^{N_\tau}\, h_2) dN_\tau - \int_0^t h_3 (-1)^{N_\tau}\, d\tau)\Big)\big]; \tag{3.5}$$

and

$$f_1 = e^t\, \mathbf{E}\big[f_{\frac{1+(-1)^{N_t}}{2}}$$

$$\cdot \exp\Big(\int_0^t \ell n(h_1 + i(-1)^{N_\tau}\, h_2) dN_\tau + \int_0^t h_3 (-1)^{N_\tau}\, d\tau\Big)\big]; \tag{3.6}$$

where N_t is a Poisson process with unit parameter. Comparing (3.3) and (3.5) we get:

$$-\int^B \int^B \left[\mathrm{Ker}(e^{tL})(\xi, \xi') F(\xi') \xi\right] \, d\xi' \, d\xi = e^t\, \mathbf{E}\big[f_{\frac{1-(-1)^{N_t}}{2}} \cdot$$

$$\cdot \exp\Big(\int_0^t \ell n(h_1 - i(-1)^{N_\tau}\, h_2) dN_\tau - \int_0^t h_3 (-1)^{N_\tau}\, d\tau\Big)\big], \tag{3.7}$$

An analogous equation can be written for $f_1(t)$.

In this way we obtain the equality between certain Berezin integrals and expectations with respect to the standard Poisson process.

To discuss the general case, let us consider the family of operators:

$$\gamma_j = \xi_j + \frac{\delta}{\delta\xi_j}\,; \qquad \bar{\gamma}_j = \frac{1}{i}\left(\xi_j - \frac{\delta}{\delta\xi_k}\right), \qquad j = 1, \ldots, n$$

which satisfy the commutation relations:

$$\{\gamma_i, \bar{\gamma}_j\} = 0, \qquad \{\gamma_i, \gamma_j\} = \{\bar{\gamma}_i, \bar{\gamma}_j\} = 2\delta_{ij}\,,$$

Further we will use the notations: letν, x, $\mu \in M_k$

$$\nu = \{\nu_1, \ldots, \nu_n\}, \quad x = \{x_1, \ldots, x_\ell\}, \quad \mu = \{\mu_1, \ldots, \mu_m\},$$

such that

$$\nu \cap \mu = \nu \cap x = s \cap \mu = \emptyset;$$

we will write:

$$\gamma^{(\nu,x,\mu)} = \prod_{i=1}^{n} \gamma_{\nu_i} \cdot \prod_{j=1}^{\ell} (\bar{\gamma}_{x_j}\, \gamma_{x_j}) \cdot \prod_{r=1}^{m} \bar{\gamma}_{\mu_r}\,. \tag{3.8}$$

Let us set:

$$L = \sum_{(\nu,x,\mu)} h_{(\nu,x,\mu)}\, \gamma^{(\nu,x,\mu)}\,.$$

where $\gamma^{(\nu,x,\mu)} \in \mathbf{C}(2k)$ and $h_{(\nu,x,\mu)} \in \mathbb{C}$.

Theorem 3.1. The kernel of the operator $\exp(tL)$ acting on $\mathbf{G}(\mathrm{k})$ is given as the limit:

$$\mathrm{Ker}(e^{tL})(\xi, \xi') = \lim_{m\to\infty} \; Q_t^m(\xi, \xi') \;\in\; \mathbf{G}(2k)\,;$$

where

$$Q_t^m(\xi, \xi') =$$

$$\int^B \exp\Big\{ -\sum_{j=1}^{m} \Big(-\frac{j\cdot t}{m} \sum_{(\nu,x,\mu)} \Big[h_{(\nu,x,\mu)} \prod_{\nu} (\xi_\nu + i\rho_\nu)\cdot$$

$$\cdot \prod_{x} (2\xi_x\rho_x + i\mathbf{1}) \prod_{\mu} \frac{1}{i}(\xi_\mu - i\rho_\mu) \Big] \Big) -$$

$$-\sum_{j}^{m}\sum_{r=1}^{n}\rho_r\left(\frac{j\cdot t}{m}\right)\left[\xi_r'\left(\frac{j\cdot t}{m}\right)+\xi_r'\left(\frac{(j-1)t}{m}\right)\right]\Big\}\mathcal{D}\rho$$

if k is even, and similar formula (see Appendix 1.) holds for k odd.

Proof. The proof is rather simple and we postpone it to Appendix 1.

In this way we can write solution of the equation (3.1) as the Berezin integral:

$$F_t(\xi)\,=\,(e^{tL}\,F_0)(\xi)\,=\,\int^B[\mathrm{Ker}(e^{tL})(\xi,\xi')F_0(\xi')]\,\mathcal{D}\xi'\,=$$

$$=\,\sum_{\mu\in M}f_\mu(t)\cdot\xi^\mu\,;$$

with $f_\mu(t)\,\in\,\mathbb{C}$.

Using Berezin integration rules we immediately get:

$$f_\mu(t)\,=\int^B[\,F_t(\xi)\,\cdot\xi^{\mu^c}]\,\mathcal{D}\xi\,=$$

$$=\int^B\left\{\int^B[\mathrm{Ker}(e^{tL})(\xi,\xi')F_0(\xi')]\,\mathcal{D}\xi'\,\cdot\xi^{\mu^c}\right\}\mathcal{D}\xi; \tag{3.9}$$

where $\mu^c\,\in\,M$ is a complementary multiindex to μ, i.e. $\mu\,=\,(\mu_1,\ldots,\mu_n)$ and $\mu^c\,=\,(\mu_1^c,\ldots,\mu_{n'}^c)$, are such that $\{\mu_1,\ldots,\mu_n\}\bigcap\{\mu_1^c,\ldots,\mu_{n'}^c\}=\emptyset$ and $\{\mu_1,\ldots,\mu_n\}\bigcup\,\{\mu_1^c,\ldots,\mu_{n'}^c\}=\{1,\,2,\ldots,\,k\}$, and we assume here that $\emptyset^c\,=\,\{1,2,...,k\}$.

Let us fix a total ordering $\prec$ on M (for example lexicographic order). It induces a total ordering on the set of monomials ξ^μ in the following way:

$$\xi^\mu\,\prec\,\xi^{\mu'}\qquad\text{if}\,\mu\,\prec\,\mu'\,;$$

let us rename all monomials with respect to the order $\prec$ by $\xi^1,\,\xi^2,\ldots,\xi^{2^k}$, which form the basis of $\mathbf{G}(k)$ as a 2^k-dimensional linear space.

In this basis the matrix elements of the operator $\gamma^{(\nu,x,\mu)}\,=\,\prod_{i=1}^n\gamma_{\nu_i}\cdot\prod_{j=1}^\ell(\bar\gamma_{x_j}\,\gamma_{x_j})\cdot\prod_{r=1}^m\bar\gamma_{\mu_r}$ are explicitely computable, so equation (3.1) could be rewritten:

$$\frac{\partial f_\alpha(t)}{\partial t}=\sum_{\beta\,=\,1}^{2^k}L_{\alpha,\,\beta}\,f_\beta(t)\,; \tag{3.10}$$

where the coefficients $L_{\alpha,\beta} \equiv L_{\alpha,\beta}(h_1, h_2, \ldots, h_{2^k}) \in \mathbb{C}$.

With an obvious definition of $\Phi(\alpha,\beta)$ and $\Psi(\alpha)$, equation (3.10) can be rewritten

$$\frac{\partial f_\alpha(t)}{\partial t} = \sum_{\beta=1}^{2^k-1} \exp[\,\Phi(\alpha,\beta)] \cdot f_{\alpha\oplus\beta}(t) + \Psi(\alpha) \cdot f_\alpha(t); \tag{3.11}$$

with the initial condition

$$f_\alpha(0) = f_\alpha;$$

and where the sign $\oplus$ stands for the sum modulo 2^k.

The solution of the linear system (3.11) is given by

$$f_\alpha(t) = e^{(2^k-1)\cdot t} \cdot \mathbf{E}\Big[f_{\alpha \oplus N_t} \cdot \exp\Big(\int_0^t \Psi(\alpha \oplus N_t \ominus N_\tau)\, d\tau + + \sum_{\beta=1}^{2^k-1} \int_0^t \Phi(\alpha \oplus N_t \ominus N_\tau \ominus \beta,\, \beta)\, d\, N_\tau^\beta \Big)\Big]; \tag{3.12}$$

where $N_t = \sum_{\beta=1}^{2^k-1} \beta N_t^\beta$ is the sum of $2^k - 1$ independent Poisson processes and $\alpha \oplus N_t$, $\alpha \oplus N_t \ominus N_\tau$, and $\alpha \oplus N_t \ominus N_\tau \ominus \beta$ stand for sums and differences modulo 2^k. (For complete proof see [DJLS].)

Now comparing (3.9) and (3.12)

$$\int^B \Big\{ \int^B [\mathrm{Ker}(e^{tL})(\xi,\xi')F_0(\xi')]\, \mathcal{D}_k\xi' \cdot \xi^{\mu^c} \Big\} \mathcal{D}_k\xi =$$

$$e^{(2^k-1)\cdot t} \cdot \mathbf{E}\Big[f_{\alpha \oplus N_t} \cdot exp\Big(\int_0^t \Psi(\alpha \oplus N_t \ominus N_\tau)\, d\tau + + \sum_{\beta=1}^{2^k-1} \int_0^t \Phi(\alpha \oplus N_t \ominus N_\tau \ominus \beta,\, \beta)\, d\, N_\tau^\beta \Big)\Big]; \tag{3.13}$$

Sums and differences modulo 2^k are slightly unconfortable to handle. In the next three sections we shall reformulate the theory in a space of functions of dichotomic variables. This allows the construction of a systematic formalism suitable for both theoretical and numerical analysis.

4. Representation of Grassmann Algebras in the Space of Functions of Dichotomic Variables (σ - Representation)

In this Section we discuss a linear bijection between Grassmann algebras and spaces of functions of dichotomic variables.

Let $\mathbf{Z}_2$ be $\{-1,\, 1\}$ with its natural Abelian group structure and $\mathbf{Z}_2^{\times k}$ be the direct product of k copies of $\mathbf{Z}_2$ which is finite commutative group with the unit element $e_k = (1,\, \ldots, 1)$.

Let us define $\mathcal{H}_k$ as the linear space of all "wave functions" $\chi(\cdot) : \mathbf{Z}_2^{\times k} \to \mathbb{C}$ of k dichotomic variables $\sigma_1, \ldots,\, \sigma_k$. It becomes a 2^k - dimensional Hilbert space when equipped with the inner product:

$$< \chi_1 \,, \chi_2 >_k \;=\; \sum_{\sigma \in \mathbf{Z}_2^{\times k}} \bar{\chi}_1(\sigma)\chi_2(\sigma)\,; \tag{4.1}$$

and could be interpreted as the space of (pure) states for a Heisenberg ferromagnet in a finite box.

All monomials of the Grassmann algebra $\mathbf{G}(k)$ can be indexed by elements of $\mathbf{Z}_2^{\times k}$ in the following way:

$$\sigma \;\mapsto\; \xi(\sigma) \;\equiv\; \xi^{\frac{1-\sigma}{2}} \;=\; \xi_1^{\frac{1-\sigma_1}{2}} \;\ldots\, \xi_k^{\frac{1-\sigma_k}{2}}\,, \qquad \sigma \;=\; (\sigma_1,\, \ldots\, ,\sigma_k) \;\in\; \mathbf{Z}_2^{\times k}, \tag{4.2}$$

and let us define now a map $\mathcal{I}$: $\mathcal{H}_k \;\to\; \mathbf{G}(k)$ by the formula:

$$\mathcal{I}(\chi) \;\equiv\; F_\chi(\xi) \;=\; \sum_{\sigma \in \mathbf{Z}_2^{\times k}} \chi(\sigma) \cdot \xi^{\frac{1-\sigma}{2}} \;\in\; \mathbf{G}(k)\,; \tag{4.3}$$

where the "wave function" $\chi(.) \in \mathcal{H}_k$.

It is easy to see that the map $\mathcal{I}$ is injective, i.e. $\mathcal{I}(\chi_1) \neq \mathcal{I}(\chi_2)$ if $\chi_1 \neq \chi_2$. Moreover, the map $\mathcal{J} \,:\, \mathbf{G}(k) \to \mathcal{H}_k$ defined by:

$$\mathcal{J}(F(\xi)) \;\equiv\; \chi_{F}(\sigma) \;=\; \triangle_k(\sigma) \int^{B} \xi^{\frac{1+\sigma}{2}} \cdot F(\xi)\mathcal{D}_k\xi\,, \tag{4.4}$$

where $\triangle_k(.) \,:\, \mathbf{Z}_2^{\times k} \;\to\; \mathbf{Z}_2$ is given by

$$\triangle_k(\sigma) \;=\; \prod_{l=1}^{k}\left(\frac{1-\sigma_l}{2} \;+\; \frac{1+\sigma_l}{2} \cdot \sigma_1 \cdots \sigma_{l-1}\right). \tag{4.5}$$

is the inverse of $\mathcal{I}$: $\mathcal{I} = \mathcal{J}^{-1}$, and, clearly, it is surjective. By this we obtain a linear bijection between Grassmann algebra $\mathbf{G}(k)$ and the space of functions of dichotomic variables $\mathcal{H}_k$, which we call σ - representation.

Next we turn to $\mathbf{C}(2k)$. To each $\widehat{A} \in \mathbf{C}(2k)$ corresponds a linear operator $A : \mathcal{H}_k \to \mathcal{H}_k$ given by formula:

$$\widehat{A}\, F(\xi) = \sum_{\sigma \in \mathbf{Z}_2^{\times k}} (A\chi)(\sigma) \cdot \xi^{\frac{1-\sigma}{2}} \,. \tag{4.6}$$

For any $\widehat{A} \in \mathbf{C}(2k)$, which is a linear combination of normally ordered products $\xi^{\frac{1-\epsilon}{2}} \cdot (\frac{\delta}{\delta\xi})^{\frac{1-\eta}{2}}$, where $\epsilon, \eta \in \mathbf{Z}_2^{\times k}$, we find its image A by computing the image of operators $\xi^{\frac{1-\epsilon}{2}} \cdot (\frac{\delta}{\delta\xi})^{\frac{1-\eta}{2}}$ which we denote by $a^{*\frac{1-\epsilon}{2}} a^{\frac{1-\eta}{2}}$.

Proposition 4.1 For all ϵ, $\eta \in \mathbf{Z}_2^{\times k}$ we have:

$$(a^{*\frac{1-\epsilon}{2}} \chi)(\sigma) = C_k(\epsilon, \sigma) \cdot \chi(\epsilon\sigma)\,, \qquad (a^{\frac{1-\eta}{2}} \chi)(\sigma) = A_k(\eta, \sigma) \cdot \chi(\eta\sigma)\,,$$

$$(a^{*\frac{1-\epsilon}{2}} a^{\frac{1-\eta}{2}} \chi)(\sigma) = N_k(\epsilon, \eta, \sigma) \cdot \chi(\epsilon\eta\sigma)\,;$$

where

$$C_k(\epsilon, \sigma) = \prod_{l=1}^{k} \left(\frac{1+\epsilon_l}{2} + \frac{1-\epsilon_l}{2} \cdot \frac{1-\sigma_l}{2} \cdot \epsilon_1 \cdots \epsilon_{l-1} \cdot \sigma_1 \cdots \sigma_{l-1} \right);$$

$$A_k(\eta, \sigma) = \prod_{l=1}^{k} \left(\frac{1+\eta_l}{2} + \frac{1-\eta_l}{2} \cdot \frac{1+\sigma_l}{2} \cdot \eta_1 \cdots \eta_{l-1} \cdot \sigma_1 \cdots \sigma_{l-1} \right);$$

$$N_k(\epsilon, \eta, \sigma) = A_k(\eta, \sigma) \cdot C_k(\epsilon, \eta\sigma)\,;$$

and where $\epsilon\sigma = \epsilon_1\sigma_1, \ldots, \epsilon_k\sigma_k$.

Proof. Immediately follows from the fact that

$$\xi^{\frac{1-\epsilon}{2}} \xi^{\frac{1-\sigma}{2}} = C(\epsilon, \epsilon\sigma) \cdot \xi^{\frac{1-\epsilon\sigma}{2}}\,; \qquad \left(\frac{\delta}{\delta\xi}\right)^{\frac{1-\eta}{2}} \xi^{\frac{1-\sigma}{2}} = A(\eta, \eta\sigma) \cdot \xi^{\frac{1-\eta\sigma}{2}}\,;$$

which could be checked by induction on k.

Corollary. If $F(\xi) = \sum_{\epsilon \in \mathbf{Z}_2^{\times k}} f(\epsilon) \xi^{\frac{1-\epsilon_l}{2}} \in \mathbf{G}(k)$, the image $F(a^*)$ of the linear operator $F(\xi) \in \mathbf{C}(2k)$ is

$$(F(a^*)\chi)(\sigma) = \sum_{\epsilon \in \mathbf{Z}_2^{\times k}} f(\epsilon) C_k(\epsilon, \sigma) \chi(\epsilon\sigma)\,, \tag{4.8}$$

and, in particular, when $F(\xi) = \sum_{1\le i<j\le 2k} A_{ij}\xi_i\xi_j$

$$\sum_{1\le i<j\le 2k} A_{ij}(a_i^* a_j^* \chi)(\sigma) =$$

$$= \sum_{(i,j)\in\Gamma(A)} A_{ij}\frac{1-\sigma_i}{2}\frac{1-\sigma_j}{2}\prod_{l=i+1}^{j-1}\sigma_l \chi(\sigma_1,\ldots,-\sigma_i,\ldots,-\sigma_j,\ldots,\sigma_{2k}), \qquad (4.9)$$

where $\Gamma(A) = \{(i,j), 1\le i<j\le 2k \ : A_{ij} \neq 0\}$.
Let A be a linear operator acting on $\mathcal{H}_k$ and $A(.,.)$ its matrix, given by

$$(A\chi)(\sigma) = \sum_{\sigma' \in \mathbf{Z}_2^{\times k}} A(\sigma,\sigma')\chi(\sigma'). \qquad (4.10)$$

For instance, from Proposition 2.1 we get that the matrix element of the operator $a^{*\frac{1-\epsilon}{2}} a^{\frac{1-\eta}{2}}$ is given by

$$A(\sigma,\sigma') = N_k(\epsilon,\eta,\sigma)\delta_{\epsilon\sigma,\eta\sigma'}\,.$$

Now from the formula (4.4) we obtain:

$$\int^B \xi^{\frac{1+\sigma}{2}} A\,\xi^{\frac{1-\sigma'}{2}}\mathcal{D}_k\xi = \triangle_k(\sigma)A(\sigma,\sigma'), \qquad \forall A \in \mathbf{C}(2k) \qquad (E.1)$$

which relates the Berezin integral $\int^B \xi^{\frac{1+\sigma}{2}}\hat{A}\,\xi^{\frac{1-\sigma'}{2}}\mathcal{D}_k\xi$ to the matrix $A(\sigma,\sigma')$. This is our first basic formula.

We end the Section by some remarks about kernels of $A \in \mathbf{C}(2k)$. We recall that to each $A \in \mathbf{C}(2k)$ corresponds a unique $\mathrm{Ker}(A)(\xi,\xi') \in \mathbf{G}^2(k)$, the kernel of the linear operator A, such that

$$AF(\xi) = \int^B \mathrm{Ker}(A)(\xi,\xi')F(\xi')\,\mathcal{D}_k\xi'\,, \qquad \forall F(\xi) \in \mathbf{G}(k). \qquad (4.12)$$

By exploiting Lemma 2.1, it is easy to see that for all positive integer k the kernel $\mathrm{Ker}(\mathbf{1}_{2k})(\xi,\xi')$ of the unit element $\mathbf{1}_{2K}$ of $\mathbf{C}(2k)$ is

$$\mathrm{Ker}(\mathbf{1}_{2k})(\xi,\xi') = \sum_{\sigma \in \mathbf{Z}_2^{\times k}} \triangle_k(\sigma)\xi^{\frac{1-\sigma}{2}}\xi'^{\frac{1+\sigma}{2}}\,, \qquad (4.13)$$

and more generally:

$$\mathrm{Ker}(A)(\xi,\xi') \,=\, \sum_{\sigma,\sigma' \in \mathbf{Z}_2^{\times k}} \triangle_k(\sigma') A(\sigma,\sigma') \xi^{\frac{1-\sigma}{2}} {\xi'}^{\frac{1+\sigma'}{2}}\,, \tag{4.14}$$

which, together with

$$A(\sigma,\sigma') \,=\, \triangle_k(\sigma) \iint^B \xi^{\frac{1+\sigma}{2}} \mathrm{Ker}(A)(\xi,\xi')\, {\xi'}^{\frac{1-\sigma}{2}} d{\xi'}_k \cdots d{\xi'}_1 d\xi_k \cdots d\xi_1\,, \tag{4.15}$$

relates the kernel $\mathrm{Ker}(A)(\xi,\xi')$ of A and the matrix $A(\sigma,\sigma')$.

5. Semigroups Associated to Poisson Processes

We now turn to semigroups of linear operators acting on the hilbert space $\mathcal{H}_k$. We shall give a probabilistic representation of the semigroup $\{\exp tL\}_{t\geq 0}$ generated by a non - trivial linear operator $L\,:\,\mathcal{H}_k \,\rightarrow\, \mathcal{H}_k$. It specializes the more general formulas introduced in [DJLS] to which we address the reader, nevertheless the Section will be self - contained.

We start with a special representation of $L \,\in\, L(\mathcal{H}_k,\,\mathcal{H}_k)$. For all $\epsilon \,=\, (\epsilon_1,\ldots,\epsilon_k) \,\in\, \mathbf{Z}_2^{\times k}$, and different from the identity : $\epsilon \,\neq\, e_k \,=\, (1,\ldots,1)$, let D^ϵ be the self - adjoint difference operator:

$$(D^\epsilon \chi)(\sigma) \,=\, \chi(\epsilon\sigma) \,-\, \chi(\sigma)\,; \tag{5.1}$$

which annihilates constants.
We observe that each $L \,\in\, L(\mathcal{H}_k,\,\mathcal{H}_k)$ admits the representation

$$L \,=\, \sum_{\epsilon\neq\epsilon_k} \lambda_\epsilon(.) D^\epsilon \,-\, V(.)\mathbf{1}\,,$$

where the functions $\lambda_\epsilon(.), V(.)\;:\;\mathbf{Z}_2^{\times k}\;\rightarrow\,\mathbb{C}$ are related to the matrix $L(.,.)$ of the operator L by formulas:

$$\text{i)}\quad \lambda_\epsilon(\sigma) \,=\, L(\sigma,\epsilon\sigma); \qquad\qquad \text{ii)}\quad V(\sigma) \,=\, -\sum_{\sigma'\in\mathbf{Z}_2^{\times k}} L(\sigma,\sigma').$$

Let $\Gamma(L)$ be the collection of $\epsilon \neq e_k$ such that $\lambda_\epsilon(.)$ is not identically vanishing and $|\Gamma(L)|$ be its cardinality. If $\Gamma(L) \neq \emptyset$ we call the operator L a difference operator.

Now let $\{N_t^\epsilon\}_{\epsilon\neq e_k}$ be a given collection of (2^k-1) independent Poisson processes of unit parameter which we assume to be left - continuous.

Theorem 5.1 (Probabilistic representation of semigroups)
Let $L = \sum_{\epsilon\in\Gamma(L)} \lambda_\epsilon(.)D^\epsilon - V(.)\mathbf{1}$ be a difference operator. Then

$$(e^{tL}\chi)(\sigma) =$$

$$= e^{t|\Gamma(L)|}\mathbf{E}\Big(\chi(\sigma(-1)^{N_t})\exp\{\sum_{\epsilon\in\Gamma(L)}\Big(\int_{[0,t)} ln\,\lambda_\epsilon(\sigma(-1)^{N_s})dN_s^\epsilon -$$

$$- \int_0^t \lambda_\epsilon(\sigma(-1)^{N_s})ds\Big) - \int_0^t V(\sigma(-1)^{N_s})ds\}\Big)$$

where $N_s = (N_s^1,\ldots,N_s^k)$ with $N_s^l = \sum_{\epsilon\in\Gamma(L)} \frac{1-\epsilon_l}{2}N_s^\epsilon$.

Remark By convention $\exp\int_{[0,t)} ln\,\lambda_\epsilon(\sigma(-1)^{N_s})dN_s^\epsilon$ vanishes if $\lambda_\epsilon(\sigma(-1)^{N_s}) = 0$ for some $0 < s < t$ such that $N_{s+}^\epsilon - N_s^\epsilon \neq 0$. We observe that it doesn't depend upon the choice of the branch of the logarithm.

Proof. We follow the strategy explained in [DJLS].
i) The r.h.s. defines a semigroup $\{P^t\}_{t\geq 0}$ of linear operators on $\mathcal{H}_K$ by the Markov property of Poisson process. In order to complete the proof we must show that the infinitesimal generator of $\{P^t\}_{t\geq 0}$ coincides with L.
ii) Since each wave function $\chi(.)$ is a linear superposition of characters $\chi_n(\sigma) = \sigma^n = \sigma_1^{n_1}\ldots\sigma_k^{n_k}$, $n = (n_1,\ldots,n_k) \in (\mathbf{Z}_2^*)^{\times k}$, $\mathbf{Z}_2^* \equiv \{0,1\}$, we can only consider the case of $\chi(.) = \chi_n(.)$ for some $n \in (\mathbf{Z}_2^*)^{\times k}$. Let $\xi_t^{\sigma,n}$ be the random variable

$$\xi_t^{\sigma,n} = \sum_{\epsilon\in\Gamma(L)} \int_{[0,t)} b_{\epsilon,n}(\sigma(-1)^{N_s})dN_s^\epsilon + \int_0^t m(\sigma(-1)^{N_s})ds\,,$$

where

$$b_{\epsilon,n}(\sigma) = i\pi\sum_{l=1}^k \frac{1-\epsilon_l}{2}n_l + ln\lambda_\epsilon(\sigma)\,;$$

$$m(\sigma) = -\sum_{\epsilon\in\Gamma(L)} \lambda_\epsilon(\sigma) - V(\sigma)\,.$$

By definition,

$$(P^t\chi_n)(\sigma) = \chi_n(\sigma)e^{t|\Gamma(L)|}\mathbf{E}(\exp\xi_t^{\sigma,n})\,.$$

iii) The stochastic differential $d\,\exp\xi_t^{\sigma,n}$ of the process $t \in [0,+\infty) \to \exp\xi_t^{\sigma,n}$ can be explicitly evaluated as explained in [DJLS]. It turns out that

$$d\,\exp\xi_t^{\sigma,n} =$$

$$= (\exp\xi_t^{\sigma,n})[m(\sigma(-1)^{N_t})dt + \sum_{\epsilon\in\Gamma(L)} (e^{b_{\epsilon,n}(\sigma(-1)^{N_t})} - 1)\, dN_t^{\epsilon}] =$$

$$= (\exp\xi_t^{\sigma,n})[-V(\sigma(-1)^{N_t})dt + \sum_{\epsilon\in\Gamma(L)} \lambda_\epsilon(\sigma(-1)^{N_t})(\epsilon^n dN_t^{\epsilon} - dt) - \sum_{\epsilon\in\Gamma(L)} dN_t^{\epsilon}]\,.$$

By taking the expectation of $d\,\exp\xi_t^{\sigma,n}$ for $t = 0$, since $\mathbf{E}(dN_t^{\epsilon}) = dt$ and $(D^{\epsilon}\chi_n)(\sigma) = (\epsilon^n - 1)\chi_n)(\sigma)$, it follows that $\frac{d}{dt}(P^t\chi_n)(\sigma)|_{t=0} = (L\chi_n)(\sigma)$. Therefore the infinitesimal generator of the semigroup $\{P^t\}_{t\geq 0}$ is exactly the difference operator L.

Example As an elementary illustration of Theorem 5.1, let $k = 1$ and $(D\chi)(\sigma) = \chi(-\sigma) - \chi(\sigma)$, then

$$(e^{tD}\chi)(\sigma) = \mathbf{E}(\chi(\sigma(-1)^{N_t}))\,, \tag{5.2}$$

where N_t is the Poisson process with unit parameter. Indeed

$$\mathbf{E}(\chi(\sigma(-1)^{N_t})) = \sum_{n=0}^{\infty} \frac{e^{-t}t^n}{n!}\chi(\sigma(-1)^n) =$$

$$= e^{-t}\{\chi(\sigma)\cosh t + \chi(-\sigma)\sinh t\} = (e^{tD}\chi)(\sigma)\,.$$

From Theorem 5.1 we obtain the matrix elements of the operator e^{tL}:

$$e^{tL}(\sigma,\sigma') =$$

$$= e^{t|\Gamma(L)|}\mathbf{E}\Big(\prod_{l=1}^{k} \frac{1 + \sigma_l\sigma'_l(-1)^{N_t^l}}{2}\exp\{\sum_{\epsilon\in\Gamma(L)} \Big(\int_{[0,t)} ln\,\lambda_\epsilon(\sigma(-1)^{N_s})dN_s^{\epsilon} -$$

$$- \int_0^t \lambda_\epsilon(\sigma(-1)^{N_s})ds\Big) - \int_0^t V(\sigma(-1)^{N_s})ds\}\Big)\,, \tag{E.2}$$

which is the second important equality and will be used in the next Section evaluating Berezin integrals.

6. Berezin Integrals and Poisson Processes in the σ - Representation

We shall consider Berezin integrals of the form:

$$\int^B \xi^n \exp(-S(\xi))\mathcal{D}_k\xi \;=\; \int^B \xi_1^{n_1}\xi_2^{n_2}\cdots\xi_k^{n_k}\exp(-S(\xi))\mathcal{D}_k\xi\,, \tag{6.1}$$

where $n = (n_1, \ldots, n_k) \in (\mathbf{Z}_2^*)^{\times k}$ and $S(\xi) \neq c\cdot\mathbf{1}_k$ is a non - trivial element of $\mathbf{G}(k)$. It could be interpreted as the "action" for the (Euclidean) "Fermi field" $\{\xi_1, \ldots, \xi_k\}$ in which case (6.1) would provide all (unnormalized) "correlation functions" or (Euclidean) "Green functions" of the field. In particular, for $n = (0, 0, \ldots, 0)$, (6.1) gives the "partition function":

$$Z[S] \;=\; \int^B \exp(-S(\xi))\mathcal{D}_k\xi\,. \tag{6.2}$$

Remark. We could easily consider more general integrals by taking for S an element of $\mathbf{C}$(2k). Integrals of the form (6.1) cover a large class of physical applications.

Let $s(.) : \mathbf{Z}_2^{\times k} \to \mathbb{C}$ be defined as

$$s(\epsilon) \;=\; \triangle_k(\epsilon)\int^B \xi^{\frac{1+\epsilon}{2}} S(\xi)\mathcal{D}_k\xi\,. \tag{6.3}$$

with $\triangle_k(\epsilon)$ as in (4.5).

Theorem 6.1 (Berezin integrals as Poisson averages) For each $k = 1, 2, \ldots,$ each non - trivial $S(\xi) \in \mathbf{G}_k$ and for all $n = (n_1, \ldots, n_k) \in (\mathbf{Z}_2^*)^{\times k}$

$$\int^B \xi_1^{n_1}\cdots\xi_k^{n_k}\exp(-S(\xi))\mathcal{D}_k\xi \;=$$

$$= \triangle_k(-(-1)^n)e^{(|\Gamma(S)|-s(e_k))}\mathbf{E}\Bigg(\prod_{l=1}^k \frac{1-(-1)^{N_1^l+n_l}}{2}\times \tag{E.3}$$

$$\times\exp\{\sum_{\epsilon\in\Gamma(S)}\int_{[0,1)} ln(-s(\epsilon)C_k(\epsilon, -(-1)^{N_s+n}))dN_s^\epsilon\}\Bigg)\,,$$

where $s(\epsilon)$ is given by (6.3).

In particular

$$Z[S] \,=\, e^{(|\Gamma(S)|-s(e_k))}\mathbf{E}\bigg(\prod_{l=1}^{k}\frac{1-(-1)^{N_1^l}}{2}\times$$

$$\times\exp\{\sum_{\epsilon\in\Gamma(S)}\int_{[0,1)} ln(-s(\epsilon)C_k(\epsilon,-(-1)^{N_s}))dN_s^{\epsilon}\}\bigg)\,. \tag{6.5}$$

Proof. Using formula (4.4) we get:

$$S(\xi) \,=\, \sum_{\epsilon\in\mathbf{Z}_2^{\times k}} s(\epsilon)\xi^{\frac{1-\epsilon}{2}}\,, \tag{6.6}$$

and, by hypothesis, the subset $\Gamma(S) \,=\, \{\epsilon\in\mathbf{Z}_2^{\times k},\epsilon\neq e_k \,:\, s(\epsilon)\neq 0\}$ is non empty. To the operator $-S(\hat{\xi})\in\mathbf{C}(2k)$ corresponds the image $L \,=\, -S(a^*)\in L(\mathcal{H}_k,\mathcal{H}_k)$ which is the difference operator:

$$L \,=\, \sum_{\epsilon\in\Gamma(S)}\lambda_\epsilon(\cdot)D^\epsilon \,-\, V(.)\mathbf{1}\,, \tag{6.7}$$

with

$$\lambda_\epsilon(\sigma) \,=\, -s(\epsilon)C_k(\epsilon,\sigma)\,, \tag{6.8}$$

$$V(\sigma) \,=\, \sum_{\epsilon\in\mathbf{Z}_2^{\times k}} s(\epsilon)C_k(\epsilon,\sigma) \,=\, s(e_k) \,-\, \sum_{\epsilon\in\Gamma(S)}\lambda_\epsilon(\sigma)\,. \tag{6.9}$$

From equations (E.1), (E.2) and equality

$$-V(\sigma) \,-\, \sum_{\epsilon\in\Gamma(S)}\lambda_\epsilon(\sigma) \,=\, -s(e_k)\,,$$

follows that

$$-\int_0^1 V(-(-1)^{N_s+n})ds \,-\, \sum_{\epsilon\in\Gamma(S)}\int_0^1\lambda_\epsilon(-(-1)^{N_s+n})ds \,=\, -s(e_k)\,.$$

$C_k(\epsilon,\sigma)$ is defined in Proposition 4.1 and N_s, N_s^l in Theorem 5.1.

We now make an important remark.

Remark . In equation (E.3) only the trajectories of the Poisson process with zero or one jump contribute to the expectation. In fact as soon as one of the factors $(1+(-1)^{N_t^r+n_r})$ in C_k vanishes the stochastic integral equals $-\infty$. On the other hand N_t^r is a sum of independent processes and the event of two processes jumping at the same instant has zero probability.

We now take advantage of the fact that the calculation of a Berezin integral has been reduced to an ordinary integral and we derive simple estimates. From Theorem (6.1) we get

$$\Big| \int^B \xi_1^{n_1} \cdots \xi_k^{n_k} \exp(-S(\xi)) \mathcal{D}_k \xi \Big| \leq$$

$$\leq e^{(|\Gamma(S)| - Res(e_k))} \mathbf{E} \Big(\prod_{l=1}^{k} \frac{1-(-1)^{N_1^l+n_l}}{2} \times$$

$$\times \exp\{ \sum_{\epsilon \in \Gamma(S)} \int_{[0,1)} ln(|s(\epsilon)| \Big| C_k(\epsilon, -(-1)^{N_s+n}) \Big|) dN_s^\epsilon \} \leq \tag{6.10}$$

$$\leq e^{(|\Gamma(S)| - Res(e_k))} \mathbf{E} \Big(\chi_{\{N_1^\epsilon = 0,1\}} \prod_{l=1}^{k} \frac{1-(-1)^{N_1^l+n_l}}{2} \prod_{\epsilon \in \Gamma(S)} |s(\epsilon)|^{N_1^\epsilon} \Big) =$$

$$e^{(-Res(e_k))} \frac{1}{2^k} \sum_{\rho_1, \ldots \rho_k = 0,1} (-1)^{\Sigma_{i=1}^k \rho_i (1+n_i)} \times \prod_{\epsilon \in \Gamma(S)} \{1 + (-1)^{\Sigma_{i=1}^k \rho_i \frac{(1-\epsilon_i)}{2}} |s(\epsilon)|\}$$

since $|C_k(\epsilon,\sigma)| \leq 1$ and $\int_{[0,1)} dN_s^\epsilon = N_1^\epsilon$; χ is the characteristic function of the event indicated.

7. Gaussian Berezin Integrals

Gaussian Berezin integrals are a particular case when the "action" S is bilinear in the "Fermi field". Let $\mathbf{G}^2(k)$ be the Grassmann algebra generated by $\{\bar\eta_1, \ldots, \bar\eta_k; \eta_1, \ldots, \eta_k\}$ and $(B)_{ij}$ any $k \times k$ matrix. Let us consider the integral

$$\int^B \bar\eta_1^{\bar\nu_1} \cdots \bar\eta_k^{\bar\nu_k} \eta_1^{\nu_1} \cdots \eta_k^{\nu_k} \exp\{-\frac{1}{2} \sum_{i,j=1}^{k} B_{ij} \bar\eta_i \eta_j\} \mathcal{D}_k \eta \mathcal{D}_k \bar\eta \, , \tag{7.1}$$

where $\bar{\nu}_l, \nu_l \in \mathbf{Z}_2^* = \{0,1\}$. Making the substitution $\xi_1 = \bar{\eta}_1, \ldots, \xi_k = \bar{\eta}_k;\ \xi_{k+1} = \eta_1, \ldots, \xi_{2k} = \eta_k$, we get

$$\sum_{i,j=1}^{k} B_{ij}\bar{\eta}_i\eta_j \;=\; \sum_{r,s=1}^{2k} A_{rs}\xi_r\xi_s\,,$$

where (A_{rs}) is the $2k \times 2k$ antisymmetric matrix defined by

$$A_{rs} \;=\; \begin{cases} \;\;2^{-1}B_{r(s-k)} \text{ when } r = 1,\ldots,k \text{ and } s = k+1,\ldots,2k; \\ -2^{-1}B_{s(r-k)} \text{ when } r = k+1,\ldots,2k \text{ and } s = 1,\ldots,k; \\ \;\;0 \qquad \text{otherwise.} \end{cases} \tag{7.2}$$

Therefore the Gaussian integral (7.1) can be written in the following form:

$$\int^{B} \bar{\eta}_1^{\bar{\nu}_1}\cdots\bar{\eta}_k^{\bar{\nu}_k}\eta_1^{\nu_1}\cdots\eta_k^{\nu_k}\exp\{-\frac{1}{2}\sum_{i,j=1}^{k} B_{ij}\bar{\eta}_i\eta_j\}\mathcal{D}_k\eta\mathcal{D}_k\bar{\eta} \;=$$

$$=\int^{B} \xi_1^{n_1}\cdots\xi_{2k}^{n_k}\exp(-\frac{1}{2}\sum_{r,s=1}^{2k} A_{rs}\xi_r\xi_s)\mathcal{D}_{2k}\xi\,, \tag{7.3}$$

and from now on we use the last form. Let $\Gamma(A)$ be the set:

$$\Gamma(A) \;=\; \{(r,s),\ 1 \le r < s \le 2k \;:\, A_{rs} \ne 0\}\,, \tag{7.4}$$

which we suppose not to be empty. From (4.9) we get that the generator L of the corresponding semigroup is given by

$$(L\psi)(\sigma) \;=$$

$$=\sum_{(r,s)\in\Gamma(A)} A_{rs}\frac{1-\sigma_r}{2}\frac{1-\sigma_s}{2}\prod_{l=r}^{s-1}\sigma_l\psi(\sigma_1,\ldots,-\sigma_r,\ldots,-\sigma_s,\ldots,\sigma_{2k})\,, \tag{7.5}$$

and therefore

$$\int^{B} \xi_1^{n_1}\cdots\xi_{2k}^{n_{2k}}\exp(-\frac{1}{2}\sum_{r,s=1}^{2k} A_{rs}\xi_r\xi_s)\mathcal{D}_{2k}\xi \;=$$

$$= \triangle_k(-(-1)^n)e^{|\Gamma(A)|}\mathbf{E}\Big(\prod_{l=1}^{2k}\frac{1-(-1)^{N_1^l+n_l}}{2}\prod_{(r,s)\in\Gamma(A)}(A_{rs})^{N_1^{(r,s)}}\times \tag{E.4}$$

$$\times \exp\{ \sum_{(r,s)\in\Gamma(A)} \int_{[0,1)} ln\Big[\frac{(1+(-1)^{N_t^r+n_r})(1+(-1)^{N_t^s+n_s})}{4} \prod_{l=r}^{s-1}(-1)^{s-r}(-1)^{N_t^l+n_l}\Big] dN_t^{(r,s)}\}\Big),$$

where $\{N_t^{(r,s)}\}_{(r,s)\in\Gamma(A)}$ is a family of independent Poisson processes with unit parameter and

$$N_t^l = \sum_{(l,m)\in\Gamma(A)} N_t^{(l,m)} + \sum_{(m,l)\in\Gamma(A)} N_t^{(m,l)} \quad \text{for all } 1\le l\le 2k.$$

Example. Let $k=1$, $n=(0,0)$ and $(A_{rs}) = i\sigma_2$ (Pauli matrix), then

$$-1 = \int^B e^{-\bar\eta\eta} d\eta d\bar\eta = \int^B e^{-\frac{1}{2}\Sigma_{r,s=1}^2 A_{rs}\xi_r\xi_s} \mathcal{D}_2\xi =$$

$$= e\mathbf{E}\Big(\frac{1-(-1)^{N_1}}{2} \exp\int_{[0,1)} ln\Big[-(-1)^{N_t}\frac{1+(-1)^{N_t}}{2}\Big] dN_t\Big) \tag{7.6}$$

This equality can be checked directly: indeed the random variable under expectation vanishes unless the Poisson process has exactly one jump for some $0<s<1$, in which case $N_1=1$, and which happens with probability e^{-1}.

Remark . Suppose we want to use our representation of Gaussian Berezin integrals in a numerical simulation. Let $2k$ be the total number of η and $\bar\eta$. The generic case will require the generation of $k(2k-1)$ independent Poisson processes. For a Dirac field on a finite d-dimensional lattice Λ the maximum number of Poisson processes involved is $L|\Lambda|(2L|\Lambda|-1))$ where L is the number of components of the field. This is a large number. However if we want to calculate for example the propagator of the free field we need much less. In fact the matrix B_{ij} which is the lattice version of the differential operator $\not\partial + \not A - M$ couples only nearest neighbours. Therefore the required number of independent Poisson processes is reduced to roughly $dL|\Lambda|$ since each site has $2d$ nearest neighbours.

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Appendix 1.

Proposition A.1. The kernel of the operator $\gamma^{(\nu,x,\mu)}$ given by (3.8) and acting on $\mathbf{G}(k)$ is given by the following formulas:

$$\mathrm{Ker}\gamma^{(\nu,x,\mu)}(\xi,\xi') = \int^{B} \prod_{j=1}^{n}(\xi_{\nu_j} + i\rho_{\nu_j}) \cdot \prod_{j=1}^{\ell}(2\xi_{x_j}\,\rho_{x_j} + i1)\cdot$$

$$\cdot \prod_{j=1}^{m} \frac{1}{i}(\xi_{\mu_j} - i\rho_{\mu_j}) \cdot \exp\big\{ - i \sum_{r=1}^{k} \rho_r(\xi_r - \xi'_r)\big\} d\rho; \qquad (A.1)$$

if k is even, and

$$\mathrm{Ker}\gamma^{(\nu,x,\mu)}(\xi,\xi') = \int^{B} \prod_{j=1}^{n}(\xi_{\nu_j} + \rho_{\nu_j}) \cdot \prod_{j=1}^{\ell}(1 - 2\xi_{x_j}\,\rho_{x_j})\cdot$$

$$\cdot \prod_{j=1}^{m} \frac{1}{i}(\xi_{\mu_j} - \rho_{\mu_j}) \cdot \exp\big\{ - \sum_{r=1}^{n} \rho_r(\xi_r + \xi'_r)\big\} d\rho; \qquad (A.2)$$

if k is odd.
The proof is straightforward, and we omit it.

Remark. Here we expressed elements of the algebra $\mathbf{G}_0(k)$ in so called "Fourier-transform" form. In fact we just multiply initial expression by ρ-monomials and then integrate over ρ.

Let us set:

$$L = \sum_{(\nu,x,\mu)} h_{(\nu,x,\mu)}\, \gamma^{(\nu,x,\mu)}\,.$$

where $\gamma^{(\nu,x,\mu)} \in \mathbf{C}(2k)$, and are of the form (3.8), and $h_{(\nu,x,\mu)} \in \mathbb{C}$.

After computations using (A.1) and (A.2) we get:

$$\mathrm{Ker}(e^{\frac{1}{n}L})(\xi,\xi') = \mathrm{Ker}(1 - \frac{1}{n}L)(\xi,\xi') + o(\frac{1}{n}) =$$

$$= \int^{B} \exp\big\{ - \frac{1}{n} \sum_{(\nu,x,\mu)} h_{(\nu,x,\mu)} \prod_{\nu}(\xi_\nu + i\rho_\nu) \prod_{x}(2\xi_x\rho_x + i1)\cdot$$

$$\cdot \prod_{\mu} \frac{1}{i}(\xi_\mu - i\rho_\mu) - \sum_{r=1}^{n} \rho_r(\xi_r + \xi'_r)\big\} d\rho_n \cdot \ldots \cdot d\rho_1 + o(\frac{1}{n}),$$

for k even and analogous formula holds for k being odd.

Remark. Since n is fixed we always understand convergence as a pointwise convergence in a finite dimensional linear space.

Finally applying Trotter's formula we have:

$$e^{tL}\, f(\xi) =$$

$$\lim_{m\to\infty}\int^{B}\dots\int^{B}\mathrm{Ker}(e^{\frac{t}{m}\,L})\left[\xi,\xi\left(\frac{(m-1)t}{m}\right)\right]\cdot$$

$$\cdot\mathrm{Ker}(e^{\frac{t}{m}\,L})\left[\xi\left(\frac{(m-1)t}{m}\right),\xi\left(\frac{(m-2)t}{m}\right)\right]\cdot\dots$$

$$\cdot\dots\mathrm{Ker}(e^{-\frac{t}{m}\,L})\left[\xi\left(\frac{t}{m}\right),\xi'\right]f(\xi)d\xi'\,d\xi\left(\frac{t}{m}\right)\cdot\dots\cdot d\xi\left(\frac{(m-1)t}{m}\right)=$$

$$=\lim_{m\to\infty}\int^{B}\dots\int^{B}\exp\left\{\sum_{j=1}^{m}\Bigg(-\frac{j\cdot t}{m}\cdot\sum_{(\nu,x,\mu)}h_{(\nu,x,\mu)}\prod_{\nu}(\xi_\nu+i\rho_\nu)\cdot\right.$$

$$\cdot\prod_{x}(2\xi_x\rho_x+i1)\cdot\prod_{\mu}\frac{1}{i}(\xi_\mu-i\rho_\mu)\Bigg)-$$

$$\left.-\sum_{j=1}^{m}\sum_{r=1}^{n}\rho_r\left(\frac{j\cdot t}{m}\right)\left(\xi_r\left(\frac{j\cdot t}{m}\right)+\xi_r\left(\frac{(j-1)t}{m}\right)\right)\right\}f(\xi')\cdot$$

$$\cdot d\rho\left(\frac{t}{m}\right)\cdot\dots\cdot d\rho(t)d\xi'd\xi\left(\frac{t}{m}\right)\cdot\dots\cdot d\xi(t)=$$

$$=\lim_{m\to\infty}\int^{B}Q_t^m(\xi,\xi')f(\xi')d\xi';\qquad (A.3)$$

and from which we get Theorem 3.1.

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